Counting Distinct 4-Dimensional Number Systems

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1 Introduction

This document provides a method to count the number of distinct 4-dimensional number systems $Y_4(F)$ over a field F based on certain invariants and properties.

2 Key Invariants and Properties

- **Characteristic of the Field (char(F))**: Determines basic algebraic properties.
- **Degree of Extension**: The dimension of the algebra over F.
- **Associativity and Commutativity**: Whether the algebra is associative, commutative, or neither.
- **Division Algebras**: Whether the algebra is a division algebra.
- **Structure Constants**: Parameters defining multiplication rules.

3 Counting Formula

The count depends on how we define the algebras and their properties:

3.1 Associative Algebras

For associative algebras, Wedderburn's theorem helps classify them:

$$N_{associative}(F) = \sum_{D \in \mathcal{D}(F)} \left(\sum_{n=1}^{4} \text{isomorphism classes of } M_n(D) \right)$$

where $\mathcal{D}(F)$ is the set of division algebras over F.

3.2 Non-Associative Algebras

For non-associative algebras, the count includes:

$$N_{non-associative}(F) = \sum_{\text{non-associative types}} \text{isomorphism classes for each type}$$

4 Example: Real Numbers $\mathbb{R}$

For $F = \mathbb{R}$, finite-dimensional associative division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

$$N_{associative}(\mathbb{R}) = 3$$

For non-associative algebras, the count varies with the specific type.

5 Conclusion

The number of distinct 4-dimensional number systems $Y_4(F)$ over a field F can be vast, influenced by the algebraic properties and classification theorems. The exact count depends on whether the algebras are associative, non-associative, division algebras, and other factors.